Even Triangulations of Planar Set of Points with Steiner Points

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Abstract

Let $P \subset \mathbb{R}^2$ be a set of n points of which k are interior points. Let us call a triangulation T of P *even* if all its vertices have even degree, and *pseudo-even* if at least the k interior vertices have even degree. (Pseudo-) Even triangulations have one nice property; their vertices can be 3-colored, see [2, 3, 4]. Since one can easily check that for some sets of points, such triangulation do not exist, we show an algorithm that constructs a set S of at most $\lfloor (k+2)/3 \rfloor$ Steiner points (extra points) along with a pseudo-even triangulation T of $P \cup S = V(T)$.

1 Introduction

Let $P \subset \mathbb{R}^2$ be a set of n points. Let us for a moment suppose that along with P, we are given a parity, even or odd, for each of its n points. Given a triangulation T of P, we say that a vertex v of T is *happy* if and only if v has a degree of the parity that was originally set for v. If a vertex is not *happy* then we will say that it is *unhappy.* The problem of finding a triangulation of P that maximizes the number of *happy* vertices has recently got some attention. In [1], Aichholzer *et al.* showed that one can always find a triangulation that makes at least roughly $2n/3$ vertices happy, and they also showed a configuration of points and parities that will make at least $n/108$ vertices *unhappy*, regardless of the chosen triangulation.

In this paper we attack a problem with the same spirit, however, we use a different paradigm to solve it since the result of Aichholzer *et al.* does not ensure in general a solution. Let $P \subset \mathbb{R}^2$ be as before and assume that $0 \leq k \leq n-3$ points are inside the convex hull $Conv(P)$ of P, *i.e.* there are k interior points. In our setting, only those k interior points will have a parity assigned and it will be the same for each one of them, namely, even. Now, we look for a triangulation that makes *all* those k interior vertices *happy.* We will call such triangulations *pseudo-even*, or simply *even* in the case that also the vertices of $Conv(P)$ happen to have even degree. It is already known that a maximal planar graph is 3-colorable if and only if it is at least pseudo-even, see [4] for this characterization and [2, 3] for a general reference on 3-colorable planar graphs. So pseudo-even triangulations have at least one interesting property and we can also see this problem as that of embedding 3-colorable planar graphs on set of points. As one can easily check that for some sets of points, a pseudo-even triangulation do not exist, we will introduce extra points, also known as *Steiner points*, and then we will consider the questions: how many Steiner points are sufficient and how many are necessary to get a pseudo-even triangulation T such that $P \subseteq V(T)$? While we still have no answer for the latter question, we will present a non-trivial solution for the former, namely, we will show an algorithm with the following properties:

(i) Its output triangulation T is pseudo-even and $V(T) = P \cup S$.

(ii) $|S| \leq \lfloor (k+2)/3 \rfloor$.

(iii) At most two Steiner points of S fall on $Conv(P)$.

Note that, as T is a pseudo-even triangulation, the Steiner points of S that are interior must also get even degree.

This paper is divided as follows: in Section 2 we show our construction and in Section 3 we close with some interesting observations.

2 Points in general position

Let us quickly recall that given a polygon $\mathcal{P}$, a vertex of $\mathcal{P}$ is called *reflex* if the internal angle is larger than 180 degrees and we will call it convex otherwise.

The main result of this section is the following:

Theorem 1 *Let $P \subset \mathbb{R}^2$ be a set of n points such that k of those points are interior points. Then we can always obtain a pseudo-even triangulation adding at most $\lfloor (k+2)/3 \rfloor$ Steiner points to P, of which at most two fall on $Conv(P)$.*

Before showing the actual construction let us give the general idea. As it was pointed out in the introduction, we can talk about 3-colorable maximal planar graphs and pseudo-even triangulation unambiguously. Therefore, our idea to get a pseudo-even triangulation is to actually embed a 3-colorable maximal planar graph on P with the help of at most $\lfloor (k+2)/3 \rfloor$ Steiner points. So we will use a 3-coloration as a measure of the correctness of our algorithm. Having defined what we will actually aim at, let us start with our construction.

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Proof. Let us fix a vertex $v \in Conv(P)$ such that v has the lowest y-coordinate among all points in P. Using v as a pivot, we will sort each interior point of P by its slope with respect to v. Let $p_1, \ldots, p_k$, be a labeling, from left to right with respect to this angular order, of the internal points of P. Let p_0, p_{k+1} be the left and right neighbors of v on $Conv(P)$ respectively.

We construct a simple polygon $\mathcal{P}$ from $P \setminus \{v\}$ as follows: we add each edge $p_i p_{i+1}$, for $0 \leq i \leq k$. We call this the lower part of $\mathcal{P}$ and we will denote it by $L(\mathcal{P})$. Also, we consider the edges of $Conv(P) \setminus \{p_0 v, p_{k+1} v\}$ and we call this the upper part of $\mathcal{P}$ and we will denote it by $U(\mathcal{P})$.

Next we will triangulate $\mathcal{P}$ as follows: we will scan $L(\mathcal{P})$ from left to right and we will consider the largest chains formed by convex vertices. Note that for each chain, the left and right endpoints must be reflex vertices of $\mathcal{P}$, see to the left in Figure 1. Now, for each chain, we will make adjacent its two endpoints and we will use its lowest convex vertex as a pivot to triangulate the resulting convex polygon in case that it has more than three vertices. These convex polygons can be thought as "ears" that can be cut from $\mathcal{P}$ on $L(\mathcal{P})$. The rest of $\mathcal{P}$, outside these "ears", can be triangulated in any way. See to the right in Figure 1. If there is no convex vertex of $\mathcal{P}$ in $L(\mathcal{P})$, then the triangulation of $\mathcal{P}$ is arbitrary.

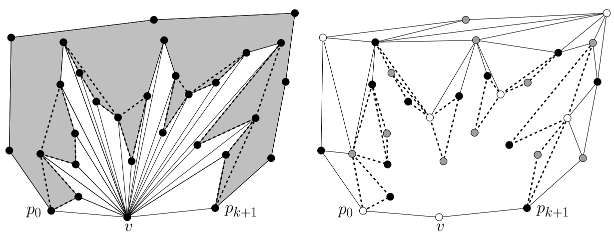

Figure 1: To the left we have the polygon $\mathcal{P}$ on $n-1$ vertices in light gray. The convex polygons formed by scanning $L(\mathcal{P})$ from left to right are shown in dashed. Note that each pair of consecutive convex polygons shares at most one vertex. To the right we see a triangulation $T(\mathcal{P})$ of $\mathcal{P}$. The dashed edges are the only ones that are not arbitrary.

Let $T(\mathcal{P})$ be the aforementioned triangulation of $\mathcal{P}$. We know that we can 3-color it, see for example [5], and note that the only point yet to be colored is v. We will show how to color v while keeping a 3-coloration of $T(\mathcal{P})$ by using Steiner points.

From this point on, our construction is done by case analysis. Note that as $T(\mathcal{P})$ is already 3-colored, if all the interior vertices of P are colored by only two colors, say $i+1, i+2$, $1 \leq i \leq 3$[1], we could use color i for v without violating the 3-coloration of $T(\mathcal{P})$, and hence, using the straight-line segments that connect v with each vertex of $L(\mathcal{P})$, we obtain a pseudo-even triangulation $T(P)$.

However, in general it is not going to happen that the interior vertices can be colored using only two colors, hence we need to do something else in such cases. We will proceed in a line-sweep fashion from p_0 to p_{k+1} with respect to the angular order given by v.

Let us fix the color of v as the color of the smallest chromatic class in the lower part $L(\mathcal{P})$ of $\mathcal{P}$ using the 3-coloration of $T(\mathcal{P})$, say that color is i without loss of generality, $1 \leq i \leq 3$. Note that the points in $L(\mathcal{P})$ with color i are the ones causing trouble to complete the desired triangulation, hence we will handle those points depending on their kind in $\mathcal{P}$, namely if they are reflex or convex vertices of $\mathcal{P}$. We will keep the invariant that, by the time we are processing an interior point p_j, *all* interior points to the left have already even degree. Also note that by this time, if the degree of p_j is odd it is because p_{j+1} has color i, otherwise we could join v and p_{j+1} and hence the conflict is somewhere to the left or even in $\mathcal{P}$, which would contradict our invariant or the valid 3-coloration of $\mathcal{P}$.

Let us start now with our case analysis, we will assume that we are currently processing the interior point p_j, $1 \leq j < k$.

(1) Point p_j of color, say $i+1$, and p_{j+1} of color i is a reflex vertex. Note that if p_{j+2} has color $i+2$, then we could introduce the edge $p_j p_{j+2}$, as p_{j+1} has already even degree in $T(\mathcal{P})$. Hence we will assume that p_j and p_{j+2} have the same color.

As p_{j+2} is of color $i+1$, then we need to complete the degree of both p_j and p_{j+1} as both degrees are odd. Here we will introduce one Steiner point s of color $i+2$ that will be adjacent, without introducing any crossing, to p_j, p_{j+1}, p_{j+2}, v, hence of even degree and we will add the straight-line segment between p_{j+2} and v, move to p_{j+2} and continue. See to the left in Figure 2

(2) Point p_j again of color $i+1$ and p_{j+1} of color i is a convex vertex. Again see that if p_{j+2} is of color $i+1$, as before, we introduce one Steiner point s of color $i+2$, and exatly the same set of adjacencies as in the case when p_{j+1} is reflex. See in the middle in Figure 2.

So let us assume that p_{j+2} has color $i+2$. Note that, as p_{j+1} is a convex vertex of $\mathcal{P}$, it must be part of one of the convex polygons we first got when $\mathcal{P}$ got triangulated, after all remember that the triangulation $T(\mathcal{P})$ is in general not arbitrary. We have now the following sub-cases:

(2.1) The vertex p_{j+1} was used as a pivot in the triangulation of $\mathcal{P}$. Consider the convex chain C formed by the points $p_j, p_{j+1}, p_{j+2}, \ldots, p_l$, where p_l is a reflex vertex. See to the right in Figure 2. Note as well that since p_{j+1} was used as a pivot, all the edges

[1] Arithmetic taken modulo three

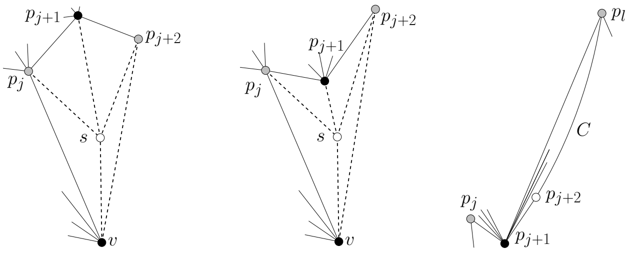

Figure 2: The point p_j is currently being processed. Point p_{j+1} is of the same color i of v. If p_j and p_{j+2} have the same color, then one Steiner point suffices to be able to move to p_{j+2}. To the right, p_j and p_{j+2} have different colors and p_{j+1} is a convex vertex that was used as a pivot to triangulate the convex polygon it is part of in $\mathcal{P}$.

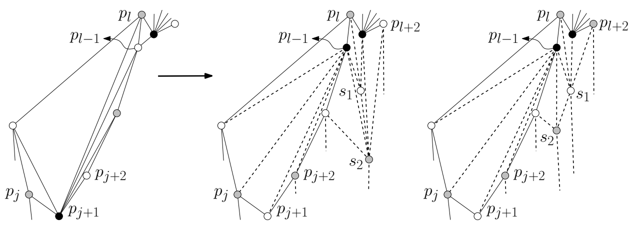

Figure 3: If p_{j+1} was used as a pivot to triangulate a convex polygon that can be cut from $\mathcal{P}$, then we can use p_{l-1} as the new pivot without changing the color of p_j or anything to its left. Note that p_l must be necessarily a reflex vertex of $\mathcal{P}$. In the middle we see the final configuration in the case that p_{l+1} is of color i and p_{l+2} is of color $i+2$. To the right we see the final configuration when p_{l+1} is of color i and p_{l+2} is of color $i+1$.

$p_{j+1}p_{j+2}, \ldots, p_{j+1}p_l$ are present.

Now we distinguish between the following cases:

(2.1.1) Point p_l is of color $i+1$, p_{l+1} is of color i and p_{l+2} is of color $i+2$.

We know that the union of all the triangles that share p_{j+1} as a vertex forms a convex polygon $\mathcal{C}$. We will change all the adjacencies inside $\mathcal{C}$ as follows:

Instead of taking p_{j+1} as the pivot that is adjacent to all vertices in $\mathcal{C}$ we will take p_{l-1}. Now we recolor p_{l-1} with color i and we will change the color of $p_{j+1}, p_{j+2}, \ldots, p_{l-1}$ to $i+2, i+1, \ldots, i+2$ respectively. Note that no other color needs to be changed.

Finally we will introduce two Steiner points s_1, s_2 of color $i+2, i+1$ respectively and we will make the following adjacencies:

(i) s_1 gets adjacent to p_{l-1}, p_l, p_{l+1} and s_2.

(ii) s_2 gets adjacent to $p_{l-2}, p_{l-1}, s_1, p_{l+1}, p_{l+2}, v$.

Additionally we introduce the edges $p_{j+1}v, \ldots, p_{l-2}v$ and $p_{l+2}v$. See to the left and in the middle of Figure 3.

Look that the previous construction can always be done without introducing any crossing. Moreover, note that with two Steiner points we complete the even degree of each point in the region $p_j, \ldots, p_{l+2}$ in which there were originally two points of color i. Thus we can move to p_{l+2} and continue.

(2.1.2) Point p_l and p_{l+1} as before and p_{l+2} is of color $i+1$.

We will proceed as before except that this time, the adjacencies of s_1, s_2 are as follows:

(i) s_1 gets adjacent to $s_2, p_{l-1}, p_l, p_{l+1}, p_{l+2}$ and v.

(ii) s_2 gets adjacent to p_{l-2}, p_{l-1}, s_1 and v.

As before, we also introduce the adjacencies $p_{j+1}v, \ldots, p_{l-2}v$ and $p_{l+2}v$. Again, every even degree is now completed and we can move to p_{l+2}. See to the right in Figure 3 for the final configuration.

(2.1.3) Point p_l as before and p_{l+1} is of color $i+2$.

Note that in this case, from p_j to p_{l+1} we are in presence of only one vertex of color i, namely p_{j+1}, thus we will introduce only one Steiner point s_1.

We will proceed as before with $\mathcal{C}$ and note that this time p_{l-1} and p_{l+1} have different colors, namely i and $i+2$ respectively. Hence the degree of p_l is already even and since p_l is a reflex vertex of $\mathcal{P}$, we can introduce the adjacency $p_{l-1}p_{l+1}$. Now we make s_1 adjacent to $p_{l-2}, p_{l-1}, p_{l+1}$ and v and finally we introduce the adjacencies $p_{j+1}v, \ldots, p_{l-2}v$ and $p_{l+1}v$.

Note that again each even degree in $p_j, \ldots, p_{l+1}$ is completed and hence we can move to p_{l+1} and continue. See to the left in Figure 4 for the final configuration.

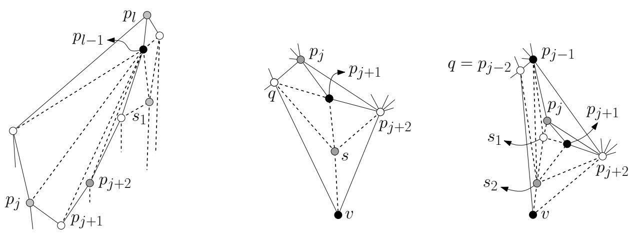

Figure 4: To the left we see the final configuration in the case that p_{j+1} was a pivot of color i and p_{l+1} is of color $i+2$. In the middle and to the right we have that, if p_{j+1} of color i was not a pivot and its neighbors have different color from each other, then one of them must necessarily be a pivot, in this case p_{j+2}. So we have to go back and remove some adjacencies that will allow us to introduce the Steiner points appropriately.

Note that the following three cases are also possible:

(2.1.4) Point p_l is of color $i+2$, p_{l+1} is of color i and p_{l+2} is of color $i+1$.

(2.1.5) Point p_l and p_{l+1} as before and p_{l+2} is of color $i+2$.

(2.1.6) Point p_l as before and p_{l+1} is of color $i+1$.

However, those cases are essentially the same as the ones explained, so we would proceed in exactly the same way but we will exchange the color of the Steiner points we are introducing. The details are left for the reader.

(2.2) In this case p_{j+1} of color i was not used as a pivot and it just takes part in a convex polygon where the pivot p_{j+2} is of color $i+2$. This restriction in colors arises from the fact that we are assuming that the triangle $p_j, p_{j+1}, p_{j+2} \in T(\mathcal{P})$ is well-colored, as explained in the beginning of case **(2)**.

Since the edge $p_j v$ is currently in the triangulation being built, there is one triangle t using it. Let $q \notin \{p_j, v\}$ be the third vertex of such a triangle t. Note that q lies to the left of the edge $p_j v$ and hence it already has even degree, moreover, the color of q is $i+2$. Now we have the following two cases:

(2.2.1) The vertex q is a Steiner point or the quadrilateral $Q = q, p_j, p_{j+1}, v$ is convex. Let us consider only the case that Q is convex, if it is not the case then q is a Steiner point and it can be moved as pleased to make Q convex without affecting anything. Thus we will flip the edge $p_j v$ for the edge qp_{j+1} and introduce one Steiner point s of color $i+1$ inside Q with its incidences to the vertices of Q, see in the middle of Figure 4.

(2.2.2) If q is not a Steiner point and Q is non-convex, then it is not hard to see that the only possible case is $q = p_{j-2}$, and p_{j-1} is a reflex vertex of $\mathcal{P}$ of color i. Note then that the edge $e = p_{j-2}p_j$ must be present in the triangulation and that p_{j-1} is adjacent to no Steiner point. Hence we will remove e and we will introduce one Steiner point s_1 of color $i+2$ that is adjacent to $p_{j-1}, p_j, p_{j+1}, s_2$, where s_2 is another new Steiner point of color $i+1$ that is adjacent to $p_{j-2}, p_{j-1}, s_1, p_{j+1}, p_{j+2}, v$. We can now move to p_{j+2} and continue. See to the right in Figure 4.

Note that the color i of v was chosen as the color of the smallest chromatic class in $L(\mathcal{P})$ and note that its cardinality can be at most $\lfloor (k+2)/3 \rfloor$. Also note that in our analysis, we assumed that the current point p_j that we are processing is neither p_0 nor p_{k+1} of $Conv(P)$. So in the case that those two extreme vertices are of color i we will introduce two Steiner points of color different that i that will subdivide the edges of $Conv(P)$ that connect p_0 and p_{k+1} with v, and hence removing any possible conflict at that stage. As we introduce one Steiner point per element of the smallest chromatic class in $L(\mathcal{P})$ the total number of Steiner points is $\lfloor (k+2)/3 \rfloor$ and the result follows.

□

3 Conclusions and Discussion

We have presented an algorithm that produces a pseudo-even triangulation adding at most $\lfloor (k+2)/3 \rfloor$ Steiner points to a given point set $P \subset \mathbb{R}^2$. It is important to note that at most two Steiner points lie on $Conv(P)$ and hence our construction keeps many extent measures of P, *i.e.* diameter, width, etc. If we do not care about modifying $Conv(P)$ or the position of the Steiner points, then *only two* Steiner points far away from $Conv(P)$ would do the job, say one at ∞ and the other at $-\infty$. Albeit being this construction possible, we do not know why it would be interesting to use it, since the output set of points does not look anything like the one that was given as the input.

For the sake of completeness it is also interesting to discuss what happens when P is in convex position and we look this time for an *even* triangulation. In [4] it was proven that if T is an even triangulation, then $|Conv(V(T))| \equiv 0 \mod 3$. The other direction can be easily proven by induction, so we do not see the necessity of writing down the details. Hence, given P in convex position, we can obtain an even triangulation T adding at most two Steiner points such that $V(T)$ remains in convex position.

We are aware that our technique could be push further to obtain a smaller number of Steiner points, probably $\lfloor (k+2)/6 \rfloor$ might be doable and we already started working out the details. Nevertheless, what it has been rather frustrating is the fact that we have not been able to come up with a lower bound on the number of Steiner points and actually, everything points to the fact that really few Steiner points might suffice, this number might even be *constant*! Finding a *simple* algorithm that uses fewer Steiner points, and finding a lower bound for this number seem interesting and challenging.

Acknowledgement

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