Colored Quadrangulations with Steiner Points

Victor Alvarez* Atsuhiro Nakamoto†

Abstract

Let P be a k-colored set of n points in general position on the plane, where $k \geq 2$. A k-colored quadrangulation of P is a *maximal* straight-edge plane graph with vertex set P satisfying the property that every interior face is a properly colored quadrilateral, *i.e.*, no edge connects vertices of the same color. It is easy to check that in general not every set of points admits a k-colored quadrangulation, and hence the use of extra points, for which we can choose the color among the k available colors, is required in order to obtain one. The extra points are known in the literature as *Steiner points*. In this paper, we show that if P satisfies some condition for the colors of the points in $Conv(P)$, then a k-colored quadrangulation of P can always be constructed using less than $\frac{(16k-2)n+7k-2}{39k-6}$ Steiner points. Our upper bound improves the previously known upper bound for $k = 3$, and represents the first bounds for $k \geq 4$.

1 Introduction

Let P be an *n-point set*, that is, a set of n points in general position on the plane. We say that P is k-colored if every point of P is colored with *exactly* one of k available colors. A quadrangulation of P is a *maximal* straight-edge plane graph with vertex set P such that every interior face is a quadrilateral. For a k-colored point set P, a quadrangulation of P is said to be k-colored if no edge of the quadrangulation connects vertices of the same color. From now on, unless stated otherwise, we will always consider P as being k-colored. Also, in our setting we are interested in having $Conv(P)$ as the outer cycle of the k-colored quadrangulations of P, thus we will always assume that any two consecutive points on $Conv(P)$ have distinct colors.

The study on k-colored quadrangulations of point sets is rather new. It is easy to see that even when $Conv(P)$ is properly colored, k-colored quadrangulations do not always exist, see [1]. Thus, if a k-colored point set P is given, and one insists on constructing a k-colored quadrangulation of P, then the use of extra points is in general needed. These extra points are known in the literature as Steiner points, and in our setting, the color of each Steiner point can be chosen from the k available colors. In [2] it was shown that for any bichromatic n-point set P, one can construct a bichromatic quadrangulation of P with the use of roughly $\frac{5n}{12}$ interior Steiner points. Those are Steiner points introduced only inside $Conv(P)$. There, it was also shown that $\frac{n}{3}$ interior Steiner points are sometimes necessary. They also considered the case when $k = 3$, and showed a surprising fact, there are 3-colored point sets that *do not* admit 3-colored quadrangulations regardless of the number of interior Steiner points used, which is definitely an unexpected result.

The strange phenomenon of not admitting 3-colored quadrangulations, even with the use of Steiner points, was recently explained in [3], where the authors showed an elegant characterization of the 3-colored point sets that admit 3-colored quadrangulations using a *finite* number of interior Steiner points. In the same paper, the authors showed that if possible, a 3-colored quadrangulation can be constructed with the use of at most $\frac{7n+17m-48}{18}$ interior Steiner points, where $|P| = n$ and $|Conv(P)| = m$. Note however that this number depends on the size of $Conv(P)$, and can get larger than wished whenever m and n are comparable in size. For example, if $m = \frac{3n}{4}$, then the bound becomes $\frac{79n-192}{72}$ which is larger than n already when $n \geq 5$.

In this paper, we show how one can use the algorithm for the bichromatic case to obtain an algorithm for a k-colored point set, for general $k \geq 3$. Our algorithm uses less than $\frac{(16k-2)n+7k-2}{39k-6}$ interior Steiner points to construct a k-colored quadrangulation of P. Our bound has the following advantages: (1) Our algorithm fully replaces the algorithm shown in [3], since it performs equally good when $Conv(P)$ is small, but it improves the worst-case behavior when $Conv(P)$ is large. For comparison, our bound for $k = 3$, at worst, is essentially $\frac{46n}{111} < \frac{5n}{12}$, while the one presented in [3] can grow larger than n if the right conditions are met. (2) Our bound represents the first bounds for the cases when $k \geq 4$.

We will divide the paper as follows: in section 2 we give the necessary definitions and the precise statement of our result, and in section 3 we prove our main Theorem.

*Fachrichtung Informatik, Universität des Saarlandes, alvarez@cs.uni-saarland.de. Partially Supported by CONACYT-DAAD of México.

†Department of Mathematics, Yokohama National University, nakamoto@ynu.ac.jp.

2 Preliminaries

In order to make this paper more self-contained, we will state the results from other papers that will be used, and will be referred to. Let us first start with some terminology. Let Q be an m-sided convex polygon, with $m \geq 4$ even, and suppose that Q is properly k-colored, where $k \geq 2$. The following are the definitions taken from [3].

Let us assume that the k chromatic classes used to color Q are $1, 2, \dots, k$, and allow us denote the color of a vertex v of Q by $c(v)$. Let us define an orientation $\mathcal{O}$ for the edges of Q as follows: if $e = uv$ is an edge of Q, then we orient e from u to v if $c(u) < c(v)$, and from v to u otherwise. Let $e^+_{\mathcal{O}}(Q)$ and $e^-_{\mathcal{O}}(Q)$ be the number of edges in clockwise and in counter-clockwise direction respectively.

Definition 1 (Winding number) *Let $\mathcal{O}$ be an orientation of Q as explained before. The winding number of Q, denoted by $\omega(Q)$, is defined as:*

$$\omega(Q) = |e^+_{\mathcal{O}}(Q) - e^-_{\mathcal{O}}(Q)|$$

for $k = 3$, and $\omega(Q) = 0$ for $k \neq 3$.

Observe that the winding number of a polygon Q is non-trivial only when Q is 3-colored.

For a point set P, we will use $\omega(P)$ as a shorthand for $\omega(Conv(P))$, extending the definition of winding number for polygons to sets of points. Finally, if P is k-colored, with $k \geq 2$, we will say that P can be *k-quadrangulated* if P admits a k-colored quadrangulation.

The following result is the one described in the introduction characterizing the 3-colored point sets which can be 3-quadrangulated with Steiner points added [3].

Theorem 1 (S.Kato, R. Mori, A. Nakamoto) *Let P be a 3-colored n-point set in general position on the plane such that $|Conv(P)| = m$. Then there exists a set $\mathbb{S}$ of Steiner points such that $P \cup \mathbb{S}$ can be 3-quadrangulated if and only if $\omega(P) = 0$. In such a case, $|\mathbb{S}| \leq \frac{7n+17m-48}{18}$.*

Now we can easily decide whether a 3-colored point set admits a 3-colored quadrangulation. Nevertheless, as we mentioned before, the number of Steiner points required by Theorem 1 can get larger than wished when m and n are comparable in size.

The main contribution of this paper is the following:

Theorem 2 *Let $k \geq 2$ be an integer, and let P be a k-colored n-point set in general position on the plane. If $\omega(P) = 0$ or $k \geq 4$, then there exists a set $\mathbb{S}$ of Steiner points such that $P \cup \mathbb{S}$ can be k-quadrangulated, and $|\mathbb{S}| < \frac{(16k-2)n+7k-2}{39k-6}$.*

Note that our Theorem, besides of being able to work with more than three chromatic classes, depends only on n and k, which is a great improvement over the previously known bound for $k = 3$.

3 Main Theorem

In order to prove our Theorem, we will need some intermediate results, the first one is easily proven using the well known Euler's formula:

Lemma 3 *Let P be an n-point set in general position on the plane where m of them lie on $Conv(P)$. Then any quadrangulation of P has $(n-1) - \frac{m}{2}$ quadrilaterals and $2(n-2) - \frac{m}{2}$ edges.*

In [3] the following Lemma was shown:

Lemma 4 (S.Kato, R. Mori, A. Nakamoto) *Let Q be a 3-colored convex polygon colored by three colors c_1, c_2, c_3. Then the winding number of Q is invariant for any bijection from $\{c_1, c_2, c_3\}$ to $\{1, 2, 3\}$.*

That is, the winding number is well-defined, and we may assume that if Q is a 3-colored convex polygon, then it is colored by $\{1, 2, 3\}$. We now have the following Lemma:

Lemma 5 *Let Q be a properly k-colored convex polygon of $m \geq 4$ sides such that $\omega(Q) = 0$. Then Q can be partitioned into $r = \frac{m-2}{2}$ properly colored quadrilaterals $Q_1, \dots, Q_r$ such that $\omega(Q_i) = 0$ for every $1 \leq i \leq r$.*

Proof. The most interesting case is when $k = 3$, which is the one we will explain here:

By Lemma 4, we may assume that the chromatic classes are exactly $\{1, 2, 3\}$. Observe that there is a vertex $v \in Q$ such that its two neighbors are of the same color. For otherwise, i.e., if every vertex of Q has two neighbors with distinct colors, then we can easily check that Q has a periodic cyclic sequence of colors $1, 2, 3$, which is contrary to $\omega(Q) = 0$. See the left in Figure 1.

Now assume that all edges of Q are oriented as explained before. Let $v \in Q$ be a vertex with two neighbors $u, w \in Q$ of the same color, where u is the right neighbor of v, and w the left neighbor. Let $x \in Q$ be the right neighbor of u. Since Q is properly colored, x has a color distinct from those of u and w, and hence we can add an edge wx to create the properly colored quadrilateral $Q_1 = xuvw$. Now, let Q' be the cycle on the convex hull of $Q \setminus \{u, v\}$. We first observe that $\omega(Q_1) = 0$ since u and w have the same color. Secondly observe that $\omega(Q') = 0$, which can be explained as follows. Since u and w have the same color, the orientations of the two edges vw and uv

are canceled in the computation of $\omega(Q)$. Moreover, the edges ux and wx are both oriented away from x. Hence we get $\omega(Q) = \omega(Q') = 0$.

We can repeat these procedures inductively on Q', as shown to the right in Figure 1. That the total number of created quadrilaterals is $\frac{m-2}{2}$ follows from Lemma 3. □

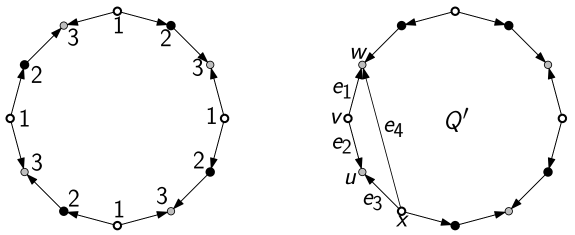

Figure 1:

The last result we need from [3] is the following:

Lemma 6 (S.Kato, R. Mori, A. Nakamoto) *Let $P = c_1 \cup c_2$ be a 2-colored n-point set in general position on the plane such that $|Conv(P)| = m$, where c_1 and c_2 are the color classes of P with $|c_1| \geq |c_2|$. Then there exists a set $\mathbb{S}$ of Steiner points such that $P \cup \mathbb{S}$ can be 2-quadrangulated, and:*

$$|\mathbb{S}| \leq \left\lfloor \frac{|c_1|}{3} \right\rfloor + \left\lfloor \frac{|c_2| - (m/2)}{2} \right\rfloor \leq \frac{5n}{12} - 1$$

The previous Lemma is essentially one of the main results of [2], and it is proven using exactly the same techniques as for Theorem 1 of [2], however, they are applied differently so the constant term on the bound of $|\mathbb{S}|$ is improved in the worst case from $(-1/3)$, in [2], to -1, in [3]. This negligible improvement of constants will play a useful role when proving Theorem 2.

The next Lemma is the last one before we proceed with the proof of Theorem 2.

Lemma 7 *Let $k \geq 2$ be an integer, and let P be a k-colored $(q+4)$-point set such that $|Conv(P)| = 4$. Then there exist two sets of Steiner points $\mathbb{S}_\Gamma$ and $\mathbb{S}_\Delta$ such that:*

- *$P \cup \mathbb{S}_\Gamma$ can be k-quadrangulated, and $|\mathbb{S}_\Gamma| \leq \frac{5q+8}{12}$.*
- *$P \cup \mathbb{S}_\Delta$ can be k-quadrangulated, and $|\mathbb{S}_\Delta| < \frac{(2k+1)q+16k}{6k}$.*

Proof. Let us divide the proof into two parts, one considering $\mathbb{S}_\Gamma$ and the other considering $\mathbb{S}_\Delta$. For simplicity, let us denote $Conv(P)$ by Q.

- Note that P can be regarded as a bichromatic point set as follows: if Q is bichromatic itself, say using colors c_1, c_2, then we can recolor every interior point of color different from c_2 with color c_1. We will rename the chromatic classes as $c_\alpha = c_1$ and $c_\beta = c_2$.

  If Q is 3-colored, say using colors c_1, c_2, c_3, then one color must appear twice on Q, say without loss of generality c_2. Proceed as before, recolor every point of color different than c_2 with a new color c_α. Rename the chromatic class c_2 as c_β.

  If Q is 4-colored, say using colors c_1, c_2, c_3, c_4, assume that c_1, c_3 and c_2, c_4 appear in diagonally opposite vertices of Q in clockwise order. Now recolor P with two new colors c_α and c_β as follows: every point of color c_2, c_4 receives color c_β. The rest of the points receive color c_α.

  As we end up having a bichromatic point set, using colors c_α, c_β, say without loss of generality that $|c_\beta| \leq |c_\alpha|$. Thus by Lemma 6 we obtain a quadrangulation of $P \cup \mathbb{S}_\Gamma$ such that:

$$|\mathbb{S}_\Gamma| \leq \left\lfloor \frac{|c_\alpha|}{3} \right\rfloor + \left\lfloor \frac{|c_\beta| - 2}{2} \right\rfloor \leq \frac{5|P|}{12} - 1 = \frac{5q+8}{12}$$

- Let us now do the following: say without loss of generality that c_1 is the smallest chromatic class among the k chromatic classes. Let us assume that Q is colored with colors other than c_1, we will see later on that this assumption only worsens the upper bound. Now let us introduce two Steiner points of color c_1 inside Q, very close to two opposite vertices of Q, and in such a way that we create a new quadrilateral Q' that is still properly colored and still contains the q interior points. Let P' be the point set formed by the vertices of Q' and the q points in its interior, see Figure 2.

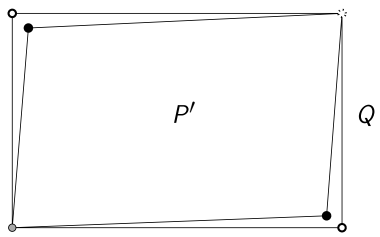

Figure 2: Points colored with color c_1 are represented in black. Quadrilateral Q' still contains the q interior points that quadrilateral Q originally contained.

Now recolor every point of P' of color different than c_1 with a new color c. This leaves only two chromatic classes, c_1 and c, where c_1 is still the smallest one. We can now proceed with the quadrangulation of a bichromatic point set again, this time obtaining:

$$\begin{aligned} |\mathbb{S}_\Delta| &\le \left\lfloor \frac{|c|}{3} \right\rfloor + \left\lfloor \frac{(|c_1|+2)-2}{2} \right\rfloor + 2 \\ &\le \frac{|c|}{3} + \frac{|c_1|}{2} + 2 = \frac{|c|+|c_1|}{3} + \frac{|c_1|}{6} + 2 \\ &< \frac{q+2}{3} + \frac{q}{6k} + 2 \\ &= \frac{q(2k+1)+16k}{6k} \end{aligned}$$

where the first inequality is obtained using Lemma 6 again. The last inequality is obtained by $|c| = q+2-|c_1|$ and the assumption that c_1 is the smallest chromatic class, so $|c_1| < \frac{q}{k}$. If $|c_1| = \frac{q}{k}$, then we have $|c_1| = \cdots = |c_k| = \frac{q}{k}$, and hence we can take c_1 so that c_1 does appear on Q. In this case, only at most one Steiner point is required in the beginning to obtain Q'. Hence we would obtain $|\mathbb{S}_\Delta| \le \frac{q(2k+1)+7k}{6k}$ which is slightly smaller, but it would still play a role reducing the bound on Theorem 2.

□

We are finally ready to prove Theorem 2:

Proof. Let P be a k-colored n-point set on the plane, where $|Conv(P)| = m$ and $q = n-m$. Then P has q interior points. If $\omega(P) = 0$, by Lemma 5, we know that we can partition $Conv(P)$ into $r = \frac{m-2}{2}$ convex quadrilaterals Q_i, $1 \le i \le r$, each of which is properly colored and has $\omega(Q_i) = 0$. If $\omega(P) \neq 0$, then by Theorem 1 the only case that makes senses is $k \ge 4$. That is, P is colored with at least four colors but only three of them appear in $Conv(P)$, causing $\omega(P) \neq 0$. In this case we cannot apply Lemma 5 directly, so we will introduce one Steiner point s inside $Conv(P)$, and very close to one vertex v of $Conv(P)$ such that s replaces v in $Conv(P)$. If the color of s is chosen such that the new $Conv(P)$ is 4-colored, and observe that this is always the case, we can proceed with Lemma 5 as before.

Let q_i be the number of interior points in quadrilateral Q_i. Using the first case of Lemma 7 on each Q_i, we get overall a set $\mathbb{S}_\Gamma = \mathbb{S}_\Gamma^1 \cup \mathbb{S}_\Gamma^2 \cup \cdots \cup \mathbb{S}_\Gamma^r$ of Steiner points, where $\mathbb{S}_\Gamma^i$ denotes the set of Steiner points used to k-quadrangulate Q_i such that:

$$\begin{aligned} |\mathbb{S}_\Gamma| &= \sum_{i=1}^{r} |\mathbb{S}_\Gamma^i| \le \sum_{i=1}^{r} \frac{5q_i+8}{12} = \frac{2r}{3} + \sum_{i=1}^{r} \frac{5q_i}{12} \\ &= \frac{m-2}{3} + \frac{5q}{12} \end{aligned}$$

Now, if we use the second case of Lemma 7 on each Q_i we get overall:

$$\begin{aligned} |\mathbb{S}_\Delta| &= \sum_{i=1}^{r} |\mathbb{S}_\Delta^i| < \sum_{i=1}^{r} \frac{(2k+1)q_i+16k}{6k} \\ &= \frac{8r}{3} + \sum_{i=1}^{r} \frac{(2k+1)q_i}{6k} \\ &= \frac{4(m-2)}{3} + \frac{(2k+1)q}{6k} \end{aligned}$$

We are assuming that in each Q_i all the chromatic classes appear. If that is not the case, say there is at least one chromatic class not appearing in some Q_j, $1 \le j \le r$, then the size of the smallest chromatic class in Q_j is 0. In such a case, as the reader can verify, we would obtain an improvement on $|\mathbb{S}_\Delta^j|$, which would clearly improve $|\mathbb{S}_\Delta|$.

Now we would like to see which one of $\mathbb{S}_\Gamma, \mathbb{S}_\Delta$ performs better, and under what circumstances. For the following, we note that $q = n-m$. If $|\mathbb{S}_\Gamma| \le |\mathbb{S}_\Delta|$, then we have:

$$\frac{m-2}{3} + \frac{5(n-m)}{12} < \frac{4(m-2)}{3} + \frac{(2k+1)(n-m)}{6k}$$

and hence $m > \frac{k(n+24)-2n}{13k-2}$. The bound on m in turn implies $q < \frac{12k(n-2)}{13k-2}$.

Let $\mathbb{S}$ be a set of Steiner points added for k-quadrangulating P, and estimate $|\mathbb{S}|$ by $\min\{|\mathbb{S}_\Gamma|, |\mathbb{S}_\Delta|\}$. Then, if $m > \frac{k(n+24)-2n}{13k-2}$, we obtain:

$$\begin{aligned} |\mathbb{S}| &\le |\mathbb{S}_\Gamma| + 1 = \frac{m-2}{3} + \frac{5q}{12} + 1 = \frac{4n+q+4}{12} \\ &< \frac{(16k-2)n+7k-2}{39k-6} \end{aligned}$$

where the second equality follows from $m = n-q$. On the other hand, if $m \le \frac{k(n+24)-2n}{13k-2}$, then

$$|\mathbb{S}| \le |\mathbb{S}_\Delta| + 1 < \frac{(16k-2)n+7k-2}{39k-6}$$

The Theorem now follows entirely. □

References

[1] C. Cortés, A. Márquez, A. Nakamoto, J. Valenzula. *Quadrangulations and 2-colorations.* 21st European Workshop on Computational Geometry, Eindhoven, March 911, 6568, 2005.

[2] V. Alvarez, T. Sakai, J. Urrutia, *Bichromatic Quadrangulations with Steiner Points*, Graph. Comb., **23**(1):85–98, 2007.

[3] S. Kato, R. Mori, A. Nakamoto, *Quadrangulations on 3-colored point sets with Steiner points and their winding numbers*, Submitted. Preliminary version in Proc. XIV Spanish Meeting on Computational Geometry, 133–136, 2011